Entanglement generation via scattering of two particles with hard–core repulsion

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Abstract

We analyse the entanglement generation in a one dimensional scattering process. The two colliding particles have a Gaussian wave function and interact by hard–core repulsion. In our analysis results on the entanglement of two mode Gaussian states are used. The produced entanglement depends in a non-obvious way on the parameters ratio of masses and initial widths. The asymptotic wave function of the two particles and its associated ellipse yield additional geometric insight into these conditions. The difference to the quantitative analysis of the amount of entanglement generated by beam splitters with squeezed light is discussed.

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I. INTRODUCTION

In the last decade the study of entanglement has become one of the major topics in the flourishing field of quantum information theory (for introductions into this topic see e. g. [1, 2]). Among other things, it has become clear that entanglement is an essential resource for many desirable operations in quantum information theory, for example quantum teleportation [3] or some protocols in quantum cryptography [4]. For a detailed understanding of entanglement it is necessary to quantify it. For pure states $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ this is fairly straightforward. We consider the reduced density operator in $\mathcal{H}_1$ (tracing out system 2)

$$\rho^1 := \mathrm{Tr}_2 \left(|\psi\rangle \langle\psi| \right) \tag{1}$$

and define the entropy of entanglement $E(|\psi\rangle)$ of the pure state $|\psi\rangle$ as the von Neumann entropy of this reduced density operator

$$E(|\psi\rangle) := S\left(\rho^1\right) \quad \text{with} \quad S(\rho) := -\mathrm{Tr}\left(\rho \, \log_2 \rho\right) \quad . \tag{2}$$

The quantity $E(|\psi\rangle)$ has the nice property that it (asymptotically) describes the number of maximally entangled singlet (or Bell) states of 2 qubits that can be obtained from many copies of the quantum state $|\psi\rangle$ when it is shared by two parties and each is allowed to operate on his part only [5]. The quantification of entanglement for mixed states is a much more difficult business which however does not concern us in this paper, as we are only dealing with pure states.

In this paper we study entanglement generation in quantum systems with a continuous degree of freedom, in particular a scattering process which generates correlations in the two-particle Schrödinger wave function. Continuous variable quantum systems are also an interesting topic in quantum information theory (cf. the recent review [6]). For instance, continuous versions of quantum teleportation [7] and quantum cryptography [8] can be found.

In particular we will be dealing with Gaussian states, since properties of these states can be often obtained in an analytic fashion such that there is also an extensive theoretical literature about these states [9, 10]. Gaussian wave functions are also a quite natural assumption. We will investigate how the amount of generated entanglement depends on the ratio of the widths of the incoming particle wave functions.

In quantum optics the entanglement production for photons can be realized experimentally and has been extensively studied, e. g. for squeezed states hitting a beam splitter [11, 12]. In contrast to that, there are only a few studies of entanglement production for scattering of two particles [13, 14]. This is surprising as scattering theory itself is a major topic in quantum mechanics. In [14] entanglement production for scattering of two Gaussian particles with hard core repulsion (cf. the potential in eq. (3)) was studied, and the present paper can be understood as a generalization of this paper which yields additional physical understanding. For, in [14] only the special case was treated, when the two particles have equal masses and equal initial widths of their wave functions. It was then found that only transient entanglement, i. e. non–vanishing entanglement during collision, can be produced. For a more general repulsive potential only a rather small quantity of asymptotic (permanent) entanglement could be obtained with the initial conditions of [14]. In [15] one can find calculations of entanglement in spin gases, where the position coordinates are treated classically.

Our calculation looks quite similar to the calculation of the entanglement that is generated in a beam splitter when at least one input state is squeezed. However, the decisive difference is that the action of the beam splitter is a *rotation* that mixes the quadrature amplitudes of different modes, whereas the scattering process is a *reflection* with respect to the non-orthogonal coordinate system given by the relative coordinates.

We now outline the organization of this paper. In section II we state the scattering process and review the exact solution of the time–dependent Schrödinger equation. In section III we calculate the asymptotic entanglement of the two particles after the scattering. Here we use extensively the covariance matrix of Gaussian states. A geometric interpretation of the results about entanglement is given in section IV where we study the wave function of the two particles after scattering. Therewith we can also understand in detail under which conditions significant entanglement is produced – and also when no entanglement at all is generated. In section V we sum up the obtained results and point out some possibilities for further research.

II. THE SCATTERING PROCESS AND ITS EXACT SOLUTION

Here we consider the quantum mechanical scattering process of two particles in one spatial dimension which interact via the potential $V(x_1 - x_2)$. As the scattering potential shall model hard–core repulsion, it is given by

$$V\left(|x_1 - x_2|\right) := \begin{cases} 0, & |x_1 - x_2| > a \\ \infty, & |x_1 - x_2| \le a \end{cases} \tag{3}$$

In contrast to [14] we consider general mass ratios of the two particles. Hence the Hamiltonian for the corresponding one dimensional scattering process is

$$H = \frac{\hat{p}_1^2}{2\,m_1} + \frac{\hat{p}_2^2}{2\,m_2} + V\left(|x_1 - x_2|\right) = \frac{\hat{p}_s^2}{2\,M_s} + \frac{\hat{p}_r^2}{2\,M_r} + V\left(|x_r|\right) \quad, \tag{4}$$

where we have defined the momenta

$$\hat{p}_s := \hat{p}_1 + \hat{p}_2,$$
$$\hat{p}_r := \mu_2\,\hat{p}_1 - \mu_1\,\hat{p}_2 \tag{5}$$

and center of mass and relative coordinate x_s and x_r, respectively,

$$x_s = \mu_1\,x_1 + \mu_2\,x_2,$$
$$x_r = x_1 - x_2 \quad, \tag{6}$$

using the mass fractions

$$\mu_1 := m_1/(m_1 + m_2), \quad \mu_2 := m_2/(m_1 + m_2) \quad. \tag{7}$$

Besides, in eq. (4) we use the masses $M_s := (m_1 + m_2)/2$ and $M_r := (m_1\,m_2)/(m_1 + m_2)$. We thus see that the Hamiltonian (4) decouples in the coordinates (x_s, x_r).

Our starting configuration at $t = 0$ are two Gaussian wave packets with widths σ_1^2 and σ_2^2. Let the mean value of particle 1 be located at $Q_1 \gg a$, that of particle 2 at $-Q_2 \ll -a$. The momentum of the first particle 1 shall be $-K$ and that of the second particle K. We first define a state $|f_0\rangle$ with wave function

$$f_0(x_1, x_2) = \phi_G\left(x_1\,;Q_1,\,-K,\,\sigma_1^2\right)\,\phi_G\left(x_2\,;-Q_2,\,K,\,\sigma_2^2\right), \tag{8}$$

where the Gaussian (located at Q and with momentum K) is defined as

$$\phi_G\left(\,;Q,\,K,\,\sigma^2\right) := \alpha(\sigma^2)\,\exp\left(i\,K\,x\right)\,\exp\left(-\frac{(x - Q)^2}{2\,\sigma^2}\right),\quad \alpha(\sigma^2) := \frac{1}{\sqrt{\sigma}\,\pi^{1/4}} \quad. \tag{9}$$

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Note that $|f_0\rangle$ itself does not define a physical initial condition for any finite Q, since the Gaussians always overlap. In a more formal approach the two wave packets have to be starting from an "infinite distance"[19].

Now we would like to find a solution $|\psi_t\rangle \in \mathcal{H}$ of the time–dependent Schrödinger equation with the Hamiltonian H in eq. (4) that coincides for $t \to -\infty$ approximatively with $|f_t\rangle :=$ $\exp(-iH_0 t)|f_0\rangle$, where

$$H_0 = \frac{p_s^2}{2\,M_s} + \frac{p_r^2}{2\,M_r} \quad . \tag{10}$$

is the free Hamiltonian. In the relative coordinate x_r we have to satisfy the following (boundary) conditions:

(A) For all $x_r \in [-a,\,a]$ we have $\psi_t(x_s,\,x_r) = 0$, $\quad \forall t \in \mathbb{R}$.

(B) The solution $\psi_t(x_s,\,x_r)$ is continuous in x_r at the boundary of the potential $x_r = a$.

(C) For $x_r \notin [-a,\,a]$ the solution $\psi_t(x_r,\,x_s)$ obeys the *free* Schrödinger equation with Hamiltonian H_0.

We now construct the solution as follows. For an arbitrary wave function $\phi(x_r)$ we introduce the unitary operator P_a that reflects the relative coordinate at $x_r = a$, i. e.

$$(P_a\,|\phi\rangle)\,(x_r) := \phi(2a - x_r) \quad . \tag{11}$$

It can be easily seen that the operator $\mathbb{1} \otimes P_a$ commutes with the free Hamiltonian (10)

$$[\mathbb{1} \otimes P_a,\ H_0] = 0 \quad . \tag{12}$$

Here it is important to keep in mind that the tensor product structure refers to relative coordinates and not to the particle Hilbert spaces. Let us define the state

$$|g_t\rangle := (\mathbb{1} \otimes P_a)\,|f_t\rangle \quad . \tag{13}$$

Because of the commutation relation (12) the evolution $t \mapsto |g_t\rangle$ is also a solution of the free Schrödinger equation. Now the difference $|f_t\rangle - |g_t\rangle$ satisfies condition (B). In order to enforce condition (A) we use a Heaviside function, and thus the general solution that we were looking for can be written as

$$\psi_t(x_s,\,x_r) = (\,f_t(x_s,\,x_r) - g_t(x_s,\,x_r)\,)\ \theta(x_r - a)\ . \tag{14}$$

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Since we have

$$\lim_{t \to -\infty} \| \, |\psi_t\rangle - |f_t\rangle \, \| = 0 \tag{15}$$

and

$$\lim_{t \to \infty} \| \, |\psi_t\rangle - |g_t\rangle \, \| = 0 \tag{16}$$

we consider the mapping $|f_t\rangle \mapsto |g_t\rangle$ as the scattering process.

It is important to note that due to the hard–core repulsion particle 1 always stays to the right of particles 2. In other words, there is only reflection and no transmission for this particular potential.

III.   ENTANGLEMENT BETWEEN THE TWO PARTICLES AFTER SCATTER-ING

In order to calculate the entanglement between the particles after the scattering, we have to consider solution (14) for times $t \to \infty$. Since eq. (16) holds and the free evolution $\exp(-iH_0 t)$ does not change the entanglement, we have

$$\lim_{t \to \infty} E(|\psi_t\rangle) = E(|g_0\rangle) \,, \tag{17}$$

where $|g_0\rangle = (\mathbb{1} \otimes P_a) \, |f_0\rangle$ (cf. eq. (13)). It is very important to note that the state $|g_0\rangle$ is a Gaussian state [9][20] For, the initial state $|f_0\rangle$ is obviously Gaussian and the reflection in relative coordinates does not change the Gaussian nature of the state. In order to obtain the entanglement of $|g_0\rangle$, we associate the following reduced density operator to the state $|g_0\rangle$

$$\rho^{\text{red}} := \text{Tr}_2 \left(|g_0\rangle \, \langle g_0| \right) \tag{18}$$

The reduced density operator of a Gaussian state is also Gaussian.

The entanglement of a Gaussian state can be determined from its covariance matrix that we now introduce. The four standard canonical operators of our two particle system are

$$\mathbf{R} := (\hat{x}_1, \, \hat{p}_1, \, \hat{x}_2, \, \hat{p}_2)^T \quad . \tag{19}$$

The 4 by 4 covariance matrix of a density operator ρ is then given as

$$\sigma_{ij} = \frac{1}{2} \text{Tr} \left(\rho \, \{R_i, \, R_j\} \right) - \text{Tr} \left(\rho \, R_i \right) \text{Tr} \left(\rho \, R_j \right) \,, \quad i, j \in \{1, \, \dots, \, 4\} \,, \tag{20}$$

where $\{ \, , \, \}$ denotes the anticommutator. We recall the following fact (cf. e. g. [9]): a linear change of canonical operators with a 4 by 4 symplectic matrix F and a displacement $\mathbf{D}$, i. e.

$$\tilde{\mathbf{R}} = F \, \mathbf{R} + \mathbf{D} \, , \qquad (21)$$

implies for the covariance matrix $\tilde{\sigma}$ of the new canonical operators $\tilde{\mathbf{R}}$

$$\tilde{\sigma} = F \, \sigma \, F^T \quad . \qquad (22)$$

In order to obtain the covariance matrix of the state

$$\rho' := |g_0\rangle \langle g_0| = U \, \rho \, U^\dagger \quad \text{with} \quad \rho := |f_0\rangle \langle f_0| \, , \quad U := \mathbb{1} \otimes P_a \quad , \qquad (23)$$

we start with the covariance matrix for $\rho = |f_0\rangle \langle f_0|$ and the standard operator coordinates $\mathbf{R}$ of eq. (19)

$$\sigma := \mathrm{Tr} \left(\mathbf{R} \, \mathbf{R}^T \rho \right) - \mathrm{Tr} \left(\mathbf{R} \, \rho \right) \, \mathrm{Tr} \left(\mathbf{R}^T \rho \right) = \begin{pmatrix} \sigma_1^2/2 & 0 & 0 & 0 \\ 0 & 1/(2 \, \sigma_1^2) & 0 & 0 \\ 0 & 0 & \sigma_2^2/2 & 0 \\ 0 & 0 & 0 & 1/(2 \, \sigma_2^2) \end{pmatrix} \, , \qquad (24)$$

as can be easily calculated from formulas (8) and (9). The covariance matrix for ρ' in eq. (23) in standard canonical coordinates $\mathbf{R}$ can be transformed as (using eq. (23))

$$\begin{aligned} \sigma'_{ij} &= \mathrm{Tr} \left(\rho' \, R_i \, R_j \right) - \mathrm{Tr} \left(\rho' \, R_i \right) \, \mathrm{Tr} \left(\rho' \, R_j \right) \\ &= \mathrm{Tr} \left(\rho \, (U^\dagger R_i \, U) \, (U^\dagger R_j \, U) \right) - \mathrm{Tr} \left(\rho \, (U^\dagger R_i \, U) \right) \, \mathrm{Tr} \left(\rho \, (U^\dagger R_j \, U) \right) \quad . \qquad (25) \end{aligned}$$

We now seek a symplectic transformation which relates the new operators $U^\dagger \mathbf{R} \, U$ to the standard operators $\mathbf{R}$. Then we can use formula (22) and obtain σ' from σ in (24). Since the operator $U = \mathbb{1} \otimes P_a$ acts on center of mass and relative coordinate, we define the vector $\mathbf{S}$ of canonical operators in this frame as

$$\mathbf{S} = (\hat{x}_s, \, \hat{p}_s, \, \hat{x}_r, \, \hat{p}_r)^T \quad . \qquad (26)$$

As can be easily seen from eqs. (6) and (5), we have

$$\mathbf{S} = F \, \mathbf{R} \quad \text{with} \quad F = \begin{pmatrix} \mu_1 & 0 & \mu_2 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & \mu_2 & 0 & -\mu_1 \end{pmatrix} \quad . \qquad (27)$$

The identity

$$U^\dagger \mathbf{R} U = F^{-1} U^\dagger \mathbf{S} U \tag{28}$$

follows immediately. As U is a reflection in the relative coordinate, we can easily calculate

$$U^\dagger \mathbf{S} U = G \mathbf{S} + \mathbf{D} \quad \text{with} \quad G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 0 \\ 0 \\ 2a \\ 0 \end{pmatrix}. \tag{29}$$

Using eq. (28) and (29), we arrive at

$$U^\dagger \mathbf{R} U = F^{-1} G F \mathbf{R} + F^{-1} \mathbf{D} \quad . \tag{30}$$

Thus the covariance matrix σ' is related to the covariance matrix σ of eq. (24) by

$$\sigma' = F^{-1} G F \sigma F^T G^T \left(F^{-1}\right)^T \quad . \tag{31}$$

We write σ' in the form

$$\sigma' = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}, \tag{32}$$

where A, B and C are 2 by 2 matrices. Then a straightforward computation of the product of matrices in eq. (31) yields

$$A = \begin{pmatrix} 2\mu_2^2 \sigma_2^2 + \frac{(\Delta\mu)^2 \sigma_1^2}{2} & 0 \\ 0 & \frac{2\mu_1^2}{\sigma_2^2} + \frac{(\Delta\mu)^2}{2\sigma_1^2} \end{pmatrix}, \quad B = \begin{pmatrix} 2\mu_1^2 \sigma_1^2 + \frac{(\Delta\mu)^2 \sigma_2^2}{2} & 0 \\ 0 & \frac{2\mu_2^2}{\sigma_1^2} + \frac{(\Delta\mu)^2}{2\sigma_2^2} \end{pmatrix},$$

$$C = \begin{pmatrix} \Delta\mu \left(\mu_1 \sigma_1^2 - \mu_2 \sigma_2^2\right) & 0 \\ 0 & \Delta\mu \left(\frac{\mu_2}{\sigma_1^2} - \frac{\mu_1}{\sigma_2^2}\right) \end{pmatrix} \quad , \tag{33}$$

where we used the abbreviation

$$\Delta\mu = \mu_1 - \mu_2 \quad . \tag{34}$$

Looking at the definitions of covariance matrix and reduced density operator, it is clear that the covariance matrix which corresponds the reduced state ρ^{red} of eq. (18) is given by the submatrix A of eq. (33). It is known [16, 17, 18] that the von Neumann entropy – as defined in eq. (2) – of a one mode Gaussian state ρ^{red} is

$$S\left(\rho^{\text{red}}\right) = \left(d + \frac{1}{2}\right) \log_2 \left(d + \frac{1}{2}\right) - \left(d - \frac{1}{2}\right) \log_2 \left(d - \frac{1}{2}\right) \quad , \tag{35}$$

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where the quantity $d \geq 1/2$ is given as

$$d^2 = \det(A) = \det(B) = 4\,\mu_1^2\,\mu_2^2 + (\Delta\mu)^2 \left[\frac{(\Delta\mu)^2}{4} + \frac{\mu_1^2\,\sigma_1^2}{\sigma_2^2} + \frac{\mu_2^2\,\sigma_2^2}{\sigma_1^2} \right] \quad . \tag{36}$$

If we insert the (positive) value of $d(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2)$ into eq. (35), we get the desired asymptotic entanglement in terms of the four parameters $\mu_1, \mu_2, \sigma_1^2, \sigma_2^2$.

Instead of calculating the von Neumann entropy of the reduced density operator one can also consider – as done in [14] – the purity of the state ρ^{red}

$$\mathcal{P}\left(\rho^{\mathrm{red}}\right) := \mathrm{Tr}\left(\left[\rho^{\mathrm{red}}\right]^2\right) \quad . \tag{37}$$

The smaller $\mathcal{P}\left(\rho^{\mathrm{red}}\right)$, the larger is the generated entanglement. It is known [9] that the purity of a Gaussian state like ρ^{red} is given by

$$\mathcal{P}\left(\rho^{\mathrm{red}}\right) = \frac{1}{2\,\sqrt{\det(A)}} = \frac{1}{2\,d} \quad . \tag{38}$$

Thus the purity is – as the von Neumann entropy - characterized by the determinant of the covariance matrix A.

We now discuss our result for the entanglement as expressed in eqs. (35) and (36). We first note that d^2 depends only on the two width ratios σ_1^2/σ_2^2, σ_2^2/σ_1^2 and on the mass fractions μ_1, μ_2 of the two particles. We see that for a large d - and therfore a large entanglement – it is necessary that one of the two width ratios is large. In contrast to that the quantities $0 < \mu_1, \mu_2 < 1$ enter in formula (36) not as a quotient, so that choosing large mass ratios does not generate a large d. We will discuss the conditions for the production of a lot of entanglement in detail in section IV.

One can easily check that always $d \geq 1/2$ and thus $S\left(\rho^{\mathrm{red}}\right) \geq 0$. If $d = 1/2$, then the entanglement vanishes according to eq. (35). No entanglement is generated for the two conditions (cf. eq. (36))

1. $\mu_1 = \mu_2 = 1/2$, i. e. the two scattered particles have equal mass. This result was already obtained in [14].

2. $\mu_1\,\sigma_1^2 = \mu_2\,\sigma_2^2$, i. e. the heavy particle has a small width, the light particle has a large width.

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IV. THE WAVE FUNCTION AFTER SCATTERING

In this section we study the wave function $g_0(x_1, x_2)$ that corresponds to the state $|g_0\rangle$ [21]. This will give us more insight into the results in eqs. (35) and (36) for the generated entanglement. In complete analogy to the calculation of the covariance matrix σ' in the last section, we can obtain the wave function of the state $|g_0\rangle = (\mathbb{1} \otimes P_a) |f_0\rangle$ in three steps from the initial wave function $f_0(x_1, x_2)$ of eq. (8):

1. Replace in $f_0(x_1, x_2)$ the coordinates (x_1, x_2) with the coordinates (x_s, x_r) using the inverse to relation (6).

2. Apply the operator $(\mathbb{1} \otimes P_a)$ by changing the variable x_r to $(2\,a - x_r)$.

3. Go back to the coordinates (x_1, x_2) with relation (6).

The result of this procedure is

$$
\begin{aligned}
g_0(x_1, x_2) \; = \; & \phi_G\left(2\,\mu_2\,x_2 + (\mu_1 - \mu_2)\,x_1;\, Q_1 - 2\,\mu_2\,a,\, -K,\, \sigma_1^2\right) \\
& \times\, \phi_G\left(2\,\mu_1\,x_1 - (\mu_1 - \mu_2)\,x_2;\, -Q_2 + 2\,\mu_1\,a,\, K,\, \sigma_2^2\right)\;,
\end{aligned}
\tag{39}
$$

where the Gaussians $\phi_G(\;)$ have been defined in eq. (9). Since the entanglement of a Gaussian state depends only on its covariance matrix, any displacement of the wave function in position or momentum space does not change its entanglement. Thus instead of $g_0(x_1, x_2)$ in eq. (39) we introduce the simpler wave function $\tilde{g}(x_1, x_2)$ that keeps only the relevant quadratic terms in x_1, x_2 in the exponent

$$
\begin{aligned}
\tilde{g}_0(x_1, x_2) \; = \; & \alpha\left(\sigma_1^2\right)\,\exp\left(-\frac{[(\mu_1 - \mu_2)\,x_1 + 2\,\mu_2\,x_2]^2}{2\,\sigma_1^2}\right) \\
& \times\, \alpha\left(\sigma_2^2\right)\,\exp\left(-\frac{[2\,\mu_1\,x_1 - (\mu_1 - \mu_2)\,x_2]^2}{2\,\sigma_2^2}\right)\;.
\end{aligned}
\tag{40}
$$

This wave function can be rewritten with a quadratic form in the exponent

$$
\tilde{g}_0(x_1, x_2) = \alpha\left(\sigma_1^2\right)\,\alpha\left(\sigma_2^2\right)\,\exp\left(-\frac{1}{2}\,\mathbf{x}^T\,L^T\,\Sigma\,L\,\mathbf{x}\right)\,,
\tag{41}
$$

where we have introduced $\mathbf{x}^T := (x_1, x_2)$ and the two matrices

$$
L := \begin{pmatrix} \mu_1 - \mu_2 & 2\,\mu_2 \\ 2\,\mu_1 & \mu_2 - \mu_1 \end{pmatrix}\,, \qquad \Sigma := \begin{pmatrix} 1/\sigma_1^2 & 0 \\ 0 & 1/\sigma_2^2 \end{pmatrix}\;.
\tag{42}
$$

Defining the matrix in the exponent of the wave function in eq. (41) as

$$M := L^T \Sigma L \quad , \tag{43}$$

a comparison with eq. (40) yields for its entries

$$M_{11} := \frac{(\mu_1 - \mu_2)^2}{\sigma_1^2} + \frac{4\,\mu_1^2}{\sigma_2^2} \,, \quad M_{22} := \frac{4\,\mu_2^2}{\sigma_1^2} + \frac{(\mu_1 - \mu_2)^2}{\sigma_2^2} \,,$$
$$M_{12} = M_{21} := 2\,(\mu_1 - \mu_2) \left(\frac{\mu_2}{\sigma_1^2} - \frac{\mu_1}{\sigma_2^2} \right) \quad . \tag{44}$$

¿From eq. (41) we see that the wave function factorizes into a product of particle 1 and 2 wave functions, if and only if $M_{12} = 0$. Eq. (44) shows us that this is the case when $\mu_1 = \mu_2$ or when $\mu_1\,\sigma_1^2 = \mu_2\,\sigma_2^2$ – in accordance with our findings in the last section.

We now discuss the two cases with factorizing wave functions in more detail. In the case $\mu_1 = \mu_2$ the widths σ_1^2, σ_2^2 of the particles are interchanged after the scattering process, as the formulas for M_{11} and M_{22} in eq. (44) show[22]. More generally, it can be shown that the two particles exchange the shape of their wave functions in the scattering process, if the two particle masses are equal. When $\mu_1\,\sigma_1^2 = \mu_2\,\sigma_2^2$, a short calculation shows that

$$M_{11} = \frac{1}{\sigma_1^2} \,, \quad M_{22} = \frac{1}{\sigma_2^2} \,, \tag{45}$$

and thus both particles keep their initial width. That no entanglement occurs for $\mu_1\,\sigma_1^2 = \mu_2\,\sigma_2^2$ depends crucially on the Gaussian shape of the incoming particle wavefunction. For, the two mixed terms proportional to $x_1\,x_2$ in the two exponents of eq. (40) have opposite sign, if $\mu_1\,\sigma_1^2 = \mu_2\,\sigma_2^2$, and therefore cancel each other.

Points (x_1, x_2) where the wave function $\tilde{g}_0(x_1, x_2)$ has the same value form an ellipse. For example, we may select those points (x_1, x_2) where the exponent in eq. (41) is $-1/2$. This leads to an analytical expression for an ellipse E_F as follows

$$E_F : \qquad \mathbf{x}^T M \,\mathbf{x} = 1 \quad . \tag{46}$$

Generally this ellipse is oblique to the x_1, x_2 axes; this is precisely then the case when entanglement occurs. The semimajor axis A and the semiminor axis B of the ellipse E_F in eq. (46) are determined from the two eigenvalues λ_1, λ_2 of M

$$A = \frac{1}{\sqrt{\lambda_2}} \,, \quad B = \frac{1}{\sqrt{\lambda_1}} \,, \quad (\lambda_1 \geq \lambda_2) \quad . \tag{47}$$
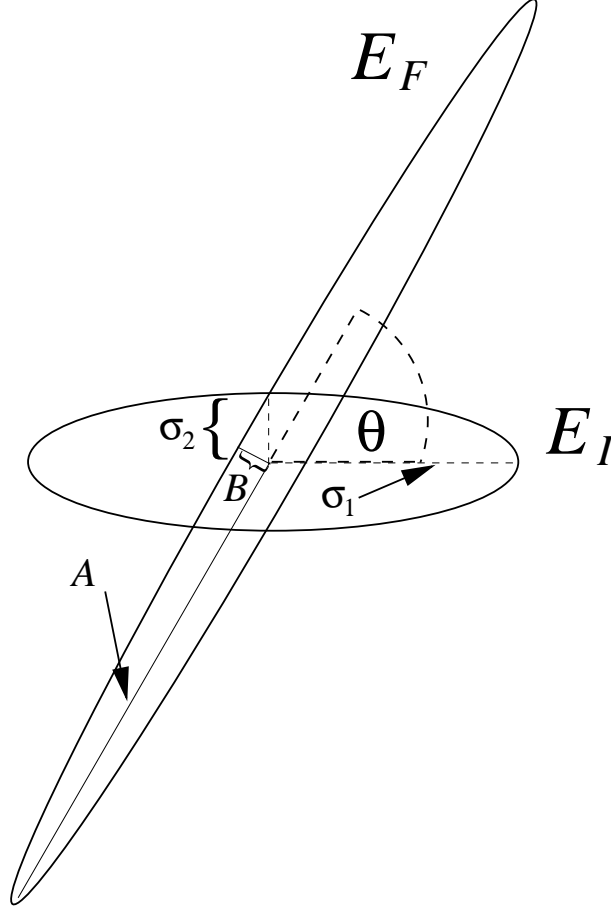
FIG. 1: Here the initial ellipse E_I with semiaxes σ_1, σ_2 and the final ellipse E_F with semiaxes A, B are depicted. The rotation angle θ of E_F is approximately $63,4^0$ and thus the length of the semimajor axis A is more than doubled in comparison to σ_1, whereas the semiminor axis B is shrunk by the same factor in comparison to σ_2.

Let θ be the angle between the semimajor axis of E_F and the x_1 axis. θ is determined by the eigenvectors of M. In fig. 1 we depict the geometric quantities A, B and θ. We can also associate to the initial wave function $f_0(x_1,\,x_2)$ in eq. (8) an ellipse by keeping – as done for the wave function $\tilde{g}_0(x_1,\,x_2)$ – only the quadratic terms in the exponent yielding

$$\tilde{f}_0(x_1,\,x_2) = \alpha\left(\sigma_1^2\right)\,\alpha\left(\sigma_2^2\right)\,\exp\left(-\frac{1}{2}\mathbf{x}^T\Sigma\,\mathbf{x}\right), \tag{48}$$

where the diagonal matrix Σ has been defined in eq. (42). The quadratic form in the exponent of equation (48) defines an ellipse E_I where the major axes are the widths σ_1, σ_2 of the two particles. We adopt the convention that $\sigma_1^2 \geq \sigma_2^2$. Thus we can understand the effect of scattering in a geometric way as a transformation of the initial ellipse E_I into the

final ellipse E_F (see fig. 1). Because of eq. (43) and $\det L = -1$,

$$\det M = \det \Sigma = \frac{1}{\sigma_1^2 \, \sigma_2^2} \; , \tag{49}$$

and thus the areas of the ellipses E_I and E_F are equal.

In general we get for the major axes of the final ellipse (cf. eq. (47)) rather complicated expressions in our parameters μ_1, μ_2, σ_1^2, σ_2^2. A significant simplification takes place, if we consider the case that one particle has a much greater width than the other one, i. e. the semimajor axis of the initial ellipse E_I is much longer than the semiminor axis ($\sigma_1 \gg \sigma_2$). This is also the most interesting case, as only then a lot of entanglement can be created via scattering, as we have remarked before. When $\sigma_1 \gg \sigma_2$, it holds approximately that the semimajor axis of the ellipse E_I is mapped via the transformation L in eq. (42) to the semimajor axis of the ellipse E_F. This means that

$$\mathbf{P} := L \begin{pmatrix} \sigma_1 \\ 0 \end{pmatrix} = \sigma_1 \begin{pmatrix} 2\,\mu_1 - 1 \\ 2\,\mu_1 \end{pmatrix} \;\; , \tag{50}$$

and the length A of the semimajor axis in E_F is

$$A \approx \|\mathbf{P}\| = \sqrt{Q(\mu_1)}\;\sigma_1 \; , \tag{51}$$

where we define the quadratic polynomial

$$Q(x) := 8\,x^2 - 4\,x + 1 \;\; . \tag{52}$$

Since the areas of the ellipse E_I and E_F are equal, we get for the length of the semiminor axis

$$B \approx \frac{\sigma_2}{\sqrt{Q(\mu_1)}} \; . \tag{53}$$

In the same approximation ($\sigma_1 \gg \sigma_2$) the angle θ of the semimajor axis (cf. fig 1) can be read off from the vector $\mathbf{P}$ in eq. (50) as [23]

$$\theta = \arctan\left(\frac{2\,\mu_1}{2\,\mu_1 - 1}\right) \;\; . \tag{54}$$

If we analyze these results, we see that for $0 < \mu_1 < 1$ the polynomial $Q(\mu_1)$ in eq. (52) has values between the minimum $1/2$ (for $\mu_1 = 1/4$) and 5 (for $\mu_1 \approx 1$). Thus according to eqs. (51) and (53) the semimajor axis A can be elongated by up to a factor $\sqrt{5}$ or shortened
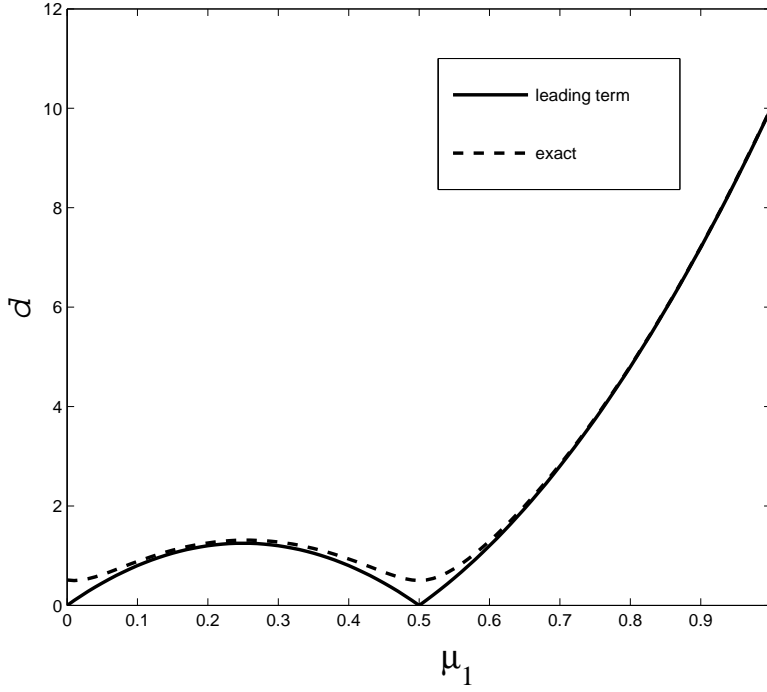
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FIG. 2: Here the quantity d is plotted as a function of μ_1 for fixed width ratio $\sigma_1/\sigma_2 = 10$. The solid line shows the leading term in formula (56), the dahed line the exact value in eq. (36).

by up to a factor $\sqrt{2}$. The semiminor axis is always scaled by the inverse factor. Looking at the angle θ in eq. (54), we find

$$
\begin{aligned}
\mu_1 > \mu_2 &\quad\Rightarrow\quad \arctan(2) < \theta < \frac{\pi}{2} , \\
\mu_1 < \mu_2 &\quad\Rightarrow\quad \frac{\pi}{2} < \theta < \pi \quad .
\end{aligned}
\tag{55}
$$

If $\mu_1 = \mu_2 = 1/2$ we have $\theta = \pi/2$ which is consistent with the generation of no entanglement. If $\mu_1 \to 0$ we get $\theta = \pi$ from eq. (54), and there is no entanglement as well. In our approximation the case $\mu_1 \to 0$ corresponds to the case $\mu_1 \sigma_1^2 = \mu_2 \sigma_2^2$ studied above, since terms of the magnitude σ_2^2/σ_1^2 have been neglected. In general, our analysis shows that the scattering process leads to more complicated transformations of the ellipse than in the situation of squeezed states hitting a beamsplitter where the initial ellipse is only rotated (cf. [12]).

How can we understand the circumstances the production of a lot of entanglement in this geometric picture? First, for $\sigma_1 \gg \sigma_2$ we calculate the leading term in σ_1/σ_2 of the quantity

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d from eq. (36)

$$d \sim |2\,\mu_1 - 1|\,\mu_1\,\frac{\sigma_1}{\sigma_2} \quad . \tag{56}$$

For the fixed value of $\sigma_1/\sigma_2 = 10$ we plot in fig. 2 the exact d and this approximation as a functions of μ_1 and find very good agreement. Thus we will use the simpler form of eq. (56) for our subsequent discussion.

According to fig. 2 (sold line) we find a local maximum of d at $\mu_1 = 1/4$ and the absolute maximum at $\mu_1 = 1$. The value at the absolute maximum is greater by a factor 8 than that at the local one. If one looks only at the angle of the final ellipse E_F, one would expect the absolute maximum for $\mu_1 = 1/4$, since for this value we get the angle $\theta = 3\,\pi/4$ from eq. (54). Here the ellipse E_F is more tilted in the x_1, x_2 plane than at the value $\mu_1 = 1$ where we obtain $\theta = \arctan(2) \approx 63, 4^0$. The explanation why the value of d for $\mu_1 = 1$ is greater than that for $\mu_1 = 1/4$ lies in the factor $\sqrt{Q(\mu_1)}$ in eqs. (51) and (53) that can lead to a stretching or shrinking of the initial ellipse E_I . For, at $\mu_1 = 1/4$ the polynomial $Q(\mu_1)$ has – as mentioned before – its minimum with value $1/2$ and thus

$$\frac{A(\mu_1 = 1/4)}{B(\mu_1 = 1/4)} = Q(\mu_1 = 1/4)\,\frac{\sigma_1}{\sigma_2} = \frac{1}{2}\,\frac{\sigma_1}{\sigma_2} \quad . \tag{57}$$

In contrast to that, for $\mu_1 = 1$ we get

$$\frac{A(\mu_1 = 1)}{B(\mu_1 = 1)} = Q(\mu_1 = 1)\,\frac{\sigma_1}{\sigma_2} = 5\,\frac{\sigma_1}{\sigma_2} \quad . \tag{58}$$

Thus the ratio between semimajor and semiminor axis is augmented by a factor 5, whereas for $\mu_1 = 1/4$ it is diminished by a factor 2. This effect outweighs the effect of the angle θ and explains why the absolute maximum is located at $\mu_1 = 1$.

V.   CONCLUSIONS

We have studied in detail the scattering of two Gaussian wave packets with hard core repulsion. The generated entanglement could be calculated analytically (cf. our central result in eqs. (35) and (36)). The special hard–core potential that we have considered yields Gaussians as asymptotic wave functions. Therefore, we could apply results about Gaussian states.

As we vary for the two particles the ratio of their masses and the ratio of their initial widths, it turns out that a great width ratio is necessary for the production of much entan-

glement in the scattering. If we vary for fixed and large width ratio the mass ratio of the two particles, maxima of entanglement production are reached,

1. If the mass of the particle with large width is much greater than that of the particle with small width. This is the absolute maximum.

2. If the mass of the particle with large width is approximately 1/3 of the mass of the particle with small width (local maximum as shown in fig. 2).

By contrast, no entanglement is generated,

1. If the two particles have equal mass.

2. If for masses and widths of the two particles the relation $\mu_1 \, \sigma_1^2 = \mu_2 \, \sigma_2^2$ holds, i. e. one particle has large width and small mass, the other particle small width and large mass.

These results about maxima and minima of entanglement are in our opinion not at all obvious, even though scattering at a hard core potential is one of the simplest scattering processes.

We have gained additional geometric understanding in our results by looking at the wave function after scattering. This Gaussian wave function defines an ellipse whose properties determine the degree of entanglement after scattering. This shows strong parallels to studies of entanglement in quantum optics where an ellipse can be defined for squeezed light hitting a beam splitter. However in contrast to the beamsplitter, the scattering process can not only rotate the ellipse but also stretch or squeeze it which is relevant for the amount of entanglement.

Our results indicate that there remains much to be done in the subject of entanglement production via scattering. Beyond the special case of hard core repulsion one should look at more realistic interaction potentials which allow for reflection and transmission of the particles. Also the modification of our results for bosons and fermions poses an interesting problem. We leave these tasks to future studies.

der Quanteninformationsverarbeitung").

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[1] W. K. Wooters, Phil. Trans. R. Soc. Lond. A **356**, 1717 (1998).

[2] S. Popescu and D. Rohrlich, in *Introduction to quantum computation and information*, edited by H.-K. Lo, S. Popescu, and T. Spiller (World Scientific, (1998)).

[3] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).

[4] A. K. Ekert, Phys. Rev. Lett **67**, 661 (1991).

[5] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, Phys. Rev. A **53**, 2046 (1996).

[6] S. L. Braunstein and P. van Loock, Rev. Mod. Phys. **77**, 513 (2005).

[7] S. L. Braunstein and H. J. Kimble, Phys. Rev. Lett. **80**, 869 (1998).

[8] M. Hillery, Phys. Rev. A **61**, 022309 (2000).

[9] A. Ferraro, S. Olivares, and M. G. A. Paris, *Gaussian states in continuous variable quantum information* (Bibliopolis, Napoli, 2005), available as quant-ph/0503237.

[10] R. Simon, E. C. G. Sudarshan, and N. Mukunda, Phsy. Rev. A **36**, 3686 (1987).

[11] C. Silberhorn, P. K. Lam, O. Weiß, F. König, N. Korolkova, and G. Leuchs, Phys. Rev. Lett. **86**, 4267 (2001).

[12] M. S. Kim, W. Son, V. Buzek, and P. L. Knight, Phys. Rev. A **65**, 032323 (2002).

[13] A. Tal and G. Kurizki, Phys. Rev. Lett **94**, 160503 (2005).

[14] C. K. Law, Phys. Rev. A **70**, 062311 (2004).

[15] J. Calsamiglia, L. Hartmann, W. Dür, and H.-J. Briegel, Phys. Rev. Lett. **95**, 180502 (2005).

[16] G. S. Agarwal, Phys. Rev. A **3**, 828 (1971).

[17] A. Serafini, F. Illuminati, and S. D. Silvia, J. Phys. B: At.Mol.Opt.Phys. **37**, L21 (2004).

[18] A. Botero and B. Reznik, Phys. Rev. A **67**, 052311 (2003).

[19] Following standard methods of scattering theory, we have to transform the state $|f_0\rangle$ backwards in time according to the free evolution $\exp(-iH_0 t)$ that is obtained from H by dropping the V-term. Then we obtain a well-defined unitary transformation for the scattering process by the following limit. Apply the backwards directed free evolution $\exp(i H_0 t)$, apply then the interacting time evolution $\exp(-2 i H t)$ and transform the result again backwards in time by $\exp(i H_0 t)$. The S matrix is then obtained in the limit $t \to \infty$.

[20] A Gaussian state has a Wigner function that is a multi–dimensional Gaussian function.

[21] Note that this in not the asymptotic wave function after the scattering. However – as explained before – the entanglement of the two particles can just as well be obtained from the state $|g_0\rangle$.

[22] Here we can neglect the fact that Gaussian wave functions broaden in free time evolution, as the scattering process normally takes place on a short time scale compared to this effect.

[23] In general this angle depends also on the values σ_1^2, σ_2^2 of the initial ellipse.